

A FIXED POINT APPROACH TO THE STABILITY OF ADDITIVE-QUADRATIC FUNCTIONAL EQUATIONS IN MODULAR SPACES

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ABSTRACT. In this paper, we prove the generalized Hyers-Ulam stability for the following additive-quadratic functional equation

$$f(2x + y) + f(2x - y) = f(x + y) + f(x - y) + 4f(x) + 2f(-x)$$

in modular spaces by using a fixed point theorem for modular spaces.

1. Introduction and preliminaries

The question of stability for a generic functional equation was originated in 1940 by Ulam [9]. Concerning a group homomorphism, Ulam posted the question asking how likely to an automorphism a function should behave in order to guarantee the existence of an automorphism near such functions. Hyers [3] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [1] for additive mappings and by Rassias [7] for linear mappings by considering an unbounded Cauchy difference, the latter of which has influenced many developments in the stability theory. This area is then referred to as the generalized Hyers-Ulam stability. A generalization of the Rassias' theorem was obtained by Găvruta [2] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

A problem that mathematicians has dealt with is "how to generalize the classical function space L^p ". A first attempt was made by Birnbaum and Orlicz in 1931. This generalization found many applications in

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differential and intergral equations with kernls of nonpower types. The more abstract generalization was given by Nakano [6] in 1950 based on replacing the particular integral form of the functional by an abstract one that satisfies some good properties. This functional was called *modular*. This idea was refined and generalized by Musielak and Orlicz [5] in 1959.

Recently, Sadeghi [8] presented a fixed point method to prove the generalized Hyers-Ulam stability of functional equations in modular spaces with the Δ_2 -condition and K. Wongkum, P. Chaipunya, and P. Kumam [10] proved the fixed point theorem and the generalized Hyers-Ulam stability for quadratic mappings in a modular space whose modular is convex, lower semi-continuous but do not satisfy the Δ_2 -condition.

In this paper, we prove the generalized Hyers-Ulam stability for the following additive-quadratic functional equation

$$(1.1) \quad f(2x + y) + f(2x - y) = f(x + y) + f(x - y) + 4f(x) + 2f(-x)$$

in modular spaces by using a fixed point theorem for modular spaces.

DEFINITION 1.1. Let X be a vector space over a field $\mathbb{K}(\mathbb{R}, \mathbb{C}, \text{ or } \mathbb{N})$.

- (1) A generalized functional $\rho : X \rightarrow [0, \infty]$ is called a *modular* if
- (M1) $\rho(x) = 0$ if and only if $x = 0$,
 - (M2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$, and
 - (M3) $\rho(z) \leq \rho(x) + \rho(y)$ whenever z is a convex combination of x and y .
- (2) If (M3) is replaced by
- (M4) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$
for all $x, y \in V$ and for all nonnegative real numbers α, β with $\alpha + \beta = 1$, then we say that ρ is *convex*.

The corresponding modular space, denoted by X_ρ , is then defined

$$X_\rho := \{x \in X \mid \rho(\lambda x) \rightarrow 0 \text{ as } \lambda \rightarrow 0\}.$$

REMARK 1.2. If a modular ρ is convex, then one has $\rho(x) \leq \delta\rho(\frac{1}{\delta}x)$ for all $x \in X_\rho$ and for all real number δ with $0 < \delta \leq 1$.

Let X_ρ be a modular space and let $\{x_n\}$ be a sequence in X_ρ . Then (i) $\{x_n\}$ is called ρ -convergent to a point $x \in X_\rho$ if $\rho(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$, (ii) $\{x_n\}$ is called ρ -Cauchy if for any $\epsilon > 0$, one has $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in \mathbb{N}$, and (iii) a subset K of X_ρ is called ρ -complete if each ρ -Cauchy sequence is ρ -convergent.

Another unnatural behavior one usually encounter is that the convergence of a sequence $\{x_n\}$ to x does not imply that $\{cx_n\}$ converges to cx for some $c \in \mathbb{K}$. Thus, many mathematicians imposed some additional

conditions for a modular to meet in order to make the multiples of $\{x_n\}$ converge naturally. Such preferences are referred to mostly under the term related to the Δ_2 -conditions.

A modular space X_ρ is said to *satisfy the Δ_2 -condition* if there exists $k \geq 2$ such that $X_\rho(2x) \leq kX_\rho(x)$ for all $x \in X$. Some authors varied the notion so that only $k > 0$ is required and called it *the Δ_2 -type condition*. In fact, one may see that these two notions coincide. There are still a number of equivalent notions related to the Δ_2 -conditions. In [4], Khamsi proved a series of fixed point theorems in modular spaces where the modulars do not satisfy Δ_2 -conditions. His results exploit one unifying hypothesis in which the boundedness of an orbit is assumed.

For a modular space X_ρ , a nonempty subset C of X_ρ , and a mapping $T : C \rightarrow C$, the orbit of T around a point $x \in C$ is the set

$$\mathbb{O}(x) := \{x, Tx, T^2x, \dots\}.$$

The quantity $\delta_\rho(x) := \sup\{\rho(u - v) \mid u, v \in \mathbb{O}\}$ is called *the orbital diameter of T at x* and if $\delta_\rho(x) < \infty$, then one says that T has a *bounded orbit at x* .

LEMMA 1.3. [4] *Let X_ρ be a modular space whose induced modular is lower semi-continuous and let $C \subseteq X_\rho$ be a ρ -complete subset. If $T : C \rightarrow C$ is a ρ -contraction, that is, there is a constant $L \in [0, 1)$ such that*

$$\rho(Tx - Ty) \leq L\rho(x - y), \quad \forall x, y \in C$$

and T has a bounded orbit at a point $x_0 \in C$, then the sequence $\{T^n x_0\}$ is ρ -convergent to a point $w \in C$.

For any modular ρ on X and any linear space V , we define a set \mathbb{M}

$$\mathbb{M} := \{g : V \rightarrow X_\rho \mid g(0) = 0\}$$

and a generalized function $\tilde{\rho}$ on \mathbb{M} by for each $g \in \mathbb{M}$,

$$\tilde{\rho}(g) := \inf\{c > 0 \mid \rho(g(x)) \leq c\phi(x, x), \quad \forall x \in V\}.$$

K. Wongkum, P. Chaipunya, and P. Kumam proved the following lemma:

LEMMA 1.4. [10] *Let V be a linear space, X_ρ a ρ -complete modular space where ρ is lower semi-continuous and convex, and $f : V \rightarrow X_\rho$ a mapping with $f(0) = 0$. Let $\phi : V^2 \rightarrow [0, \infty)$ be a mapping such that*

$$\lim_{n \rightarrow \infty} \frac{\phi(2^n x, 2^n y)}{4^n} = 0, \quad \phi(2x, 2y) \leq 2L\phi(x, y)$$

for all $x, y \in X$. Then (i) \mathbb{M} is a linear space, (ii) $\tilde{\rho}$ is a convex modular on \mathbb{M} , (iii) $\mathbb{M}_{\tilde{\rho}} = \mathbb{M}$, (iv) $\mathbb{M}_{\tilde{\rho}}$ is $\tilde{\rho}$ -complete, and (v) $\tilde{\rho}$ is lower semi-continuous.

2. The generalized Hyers-Ulam stability for (1.1) in modular spaces

Throughout this section, we assume that every modular is lower semi-continuous and convex. In this section, we prove the generalized Hyers-Ulam stability for (1.1). We start with the following theorem.

For any $f : X \rightarrow Y$, let

$$f_o(x) = \frac{1}{2}(f(x) - f(-x)), \quad f_e(x) = \frac{1}{2}(f(x) + f(-x)).$$

THEOREM 2.1. *A mapping $f : X \rightarrow Y$ satisfies (1.1) if and only if f is an additive-quadratic mapping.*

Proof. Suppose that $f : X \rightarrow Y$ satisfies (1.1). By (1.1), we have

$$f_e(2x + y) + f_e(2x - y) = f_e(x + y) + f_e(x - y) + 6f_e(x)$$

for all $x, y \in X$ and clearly, f_e is a quadratic mapping. By (1.1), we have

$$(2.1) \quad f_o(2x + y) + f_o(2x - y) = f_o(x + y) + f_o(x - y) + 2f_o(x)$$

for all $x, y \in X$. Replacing y by $x + y$ in (2.1), we get

$$(2.2) \quad f_o(3x + y) + f_o(x - y) = f_o(2x + y) - f_o(y) + 2f_o(x)$$

for all $x, y \in X$ and replacing y by $-y$ in (2.2), we get

$$(2.3) \quad f_o(3x - y) + f_o(x + y) = f_o(2x - y) + f_o(y) + 2f_o(x)$$

for all $x, y \in X$. By (2.1), (2.2), and (2.3), we obtain

$$(2.4) \quad f_o(3x + y) + f_o(3x - y) = 6f_o(x)$$

for all $x, y \in X$ and letting $y = 0$ in (2.4), we obtain

$$(2.5) \quad f_o(3x) = 3f_o(x)$$

for all $x \in X$. By (2.4) and (2.5), we have

$$f_o(x + y) + f_o(x - y) = 2f_o(x)$$

for all $x \in X$ and hence f_o is an additive mapping. Thus f is an additive-quadratic mapping. The converse is trivial. \square

THEOREM 2.2. *Let V be a linear space, X_ρ a ρ -complete modular space and $f : V \rightarrow X_\rho$ a mapping with $f(0) = 0$. Let $\phi : V^2 \rightarrow [0, \infty)$ be a mapping such that*

$$(2.6) \quad \phi(2x, 2y) \leq 2L\phi(x, y)$$

for all $x, y \in V$ and some L with $0 \leq L < 1$ and

$$(2.7) \quad \rho(f(2x+y) + f(2x-y) - f(x+y) - f(x-y) - 4f(x) - 2f(-x)) \leq \phi(x, y)$$

for all $x, y \in V$. Then there exists a unique additive-quadratic mapping $F : V \rightarrow X_\rho$ such that

$$(2.8) \quad \rho(F(x) - \frac{1}{2}f(x)) \leq \frac{3 - 2L}{8(1 - L)(2 - L)}\psi(x, 0)$$

for all $x \in V$, where $\psi(x, y) = \frac{1}{2}(\phi(x, y) + \phi(-x, -y))$.

Proof. Define a map $\tilde{\rho}$ on $\mathbb{M} = \{g : V \rightarrow X_\rho \mid g(0) = 0\}$ by

$$\tilde{\rho}(g) := \inf\{c > 0 \mid \rho(g(x)) \leq c\psi(x, 0), \forall x \in V\}$$

for each $g \in \mathbb{M}$. Similar to the proof of Lemma 1.4, we can show that $\tilde{\rho}$ satisfies (ii), (iii), (iv), and (v) in Lemma 1.4.

Define $T_o : \mathbb{M}_{\tilde{\rho}} \rightarrow \mathbb{M}_{\tilde{\rho}}$ by $T_o g(x) = \frac{1}{2}g(2x)$ for all $g \in \mathbb{M}_{\tilde{\rho}}$ and all $x \in V$. Let $g, h \in \mathbb{M}_{\tilde{\rho}}$. Suppose that $\tilde{\rho}(g - h) \leq c$ for some non-negative real number c . Then by Remark 1.2, we have

$$\rho(T_o g(x) - T_o h(x)) \leq \frac{1}{2}\rho(g(2x) - h(2x)) \leq Lc\psi(x, 0)$$

for all $x \in V$ and so $\tilde{\rho}(T_o g - T_o h) \leq L\tilde{\rho}(g - h)$. Hence T_o is a $\tilde{\rho}$ -contraction.

Now, we claim that T_o has a bounded orbit at $\frac{1}{2}f_o$. Since f_o is an odd mapping and ρ is convex, (2.7) implies that

$$(2.9) \quad \rho(f_o(2x+y) + f_o(2x-y) - f_o(x+y) - f_o(x-y) - 2f_o(x)) \leq \psi(x, y)$$

for all $x, y \in V$. Letting $y = 0$ in (2.9), we get

$$\rho(2f_o(2x) - 4f_o(x)) \leq \psi(x, 0)$$

for all $x \in V$ and so

$$\rho(\frac{1}{2}f_o(2x) - f_o(x)) \leq \frac{1}{2^2}\psi(x, 0)$$

for all $x \in V$. For any non-negative integer n , we obtain

$$\begin{aligned}
& \rho\left(\frac{1}{2^n}f_o(2^n x) - f_o(x)\right) \\
&= \rho\left(\frac{1}{2}\left[\frac{1}{2^{n-1}}f_o(2^{n-1}(2x)) - f_o(2x)\right] + \frac{1}{2}[f_o(2x) - 2f_o(x)]\right) \\
&\leq \frac{1}{2}\rho\left(\frac{1}{2^{n-1}}f_o(2^{n-1}(2x)) - f_o(2x)\right) + \frac{1}{2}\rho(f_o(2x) - 2f_o(x)) \\
&\leq \frac{1}{2}\rho\left(\frac{1}{2^{n-1}}f_o(2^{n-1}(2x)) - f_o(2x)\right) + \frac{1}{2^2}\psi(x, 0)
\end{aligned}$$

for all $x \in V$ and by induction, we have

$$(2.10) \quad \rho\left(\frac{1}{2^n}f_o(2^n x) - f_o(x)\right) \leq \sum_{i=0}^{n-1} \frac{1}{2^{i+2}}\psi(2^i x, 0)$$

for all $x \in V$ and for all non-negative integer n . Hence for any $n, m \in \mathbb{N}$, by (2.10), we get

$$\begin{aligned}
& \rho\left(\frac{1}{2^{n+1}}f_o(2^n x) - \frac{1}{2^{m+1}}f_o(2^m x)\right) \\
&\leq \frac{1}{2}\rho\left(\frac{1}{2^n}f_o(2^n x) - f_o(x)\right) + \frac{1}{2}\rho\left(\frac{1}{2^m}f_o(2^m x) - f_o(x)\right) \\
&\leq \sum_{i=0}^{n-1} \frac{1}{2^{i+3}}\psi(2^i x, 0) + \sum_{i=0}^{m-1} \frac{1}{2^{i+3}}\psi(2^i x, 0) \leq \frac{1}{4(1-L)}\psi(x, 0)
\end{aligned}$$

for all $x \in V$ and thus

$$\tilde{\rho}\left(T_o^n \frac{1}{2}f_o - T_o^m \frac{1}{2}f_o\right) \leq \frac{1}{4(1-L)}$$

for all $x \in V$. Hence T_o has a bounded orbit at $\frac{1}{2}f_o$. By Lemma 1.3, there is an $A \in \mathbb{M}_{\tilde{\rho}}$ such that $\{T_o^n \frac{1}{2}f_o\}$ $\tilde{\rho}$ -converges to A . Since $\tilde{\rho}$ is lower semi-continuous, we get

$$\tilde{\rho}(T_o A - A) \leq \liminf_{n \rightarrow \infty} \tilde{\rho}(T_o A - T_o^{n+1} \frac{1}{2}f_o) \leq \liminf_{n \rightarrow \infty} L\tilde{\rho}(A - T_o^n \frac{1}{2}f_o) = 0$$

and hence A is a fixed point of T_o in $\mathbb{M}_{\tilde{\rho}}$. Replacing x and y by $2^n x$ and $2^n y$ in (2.9), respectively, we have

$$\begin{aligned}
(2.11) \quad & \rho\left(\frac{1}{2^{n+1}}[f_o(2^n(2x+y)) + f_o(2^n(2x-y)) - f_o(2^n(x+y))\right. \\
& \left. - f_o(2^n(x-y)) - 2f_o(2^n x)]\right) \leq \frac{1}{2^{n+1}}\psi(2^n x, 2^n y) \leq \frac{L^n}{2}\psi(x, y)
\end{aligned}$$

for all $x, y \in V$. Since ρ is lower semi-continuous, by (2.11), we get

$$(2.12) \quad A(2x+y) + A(2x-y) - A(x+y) - A(x-y) - 2A(x) = 0$$

for all $x, y \in V$. Since ρ is lower semi-continuous, by (2.10), we get

$$\rho(2A(x) - f_o(x)) \leq \frac{1}{4(1-L)}\psi(x, 0)$$

for all $x \in X$ and so we have

$$(2.13) \quad \tilde{\rho}(2A - f_o) \leq \frac{1}{4(1-L)}.$$

Define $T_e : \mathbb{M}_{\tilde{\rho}} \rightarrow \mathbb{M}_{\tilde{\rho}}$ by $T_e g(x) = \frac{1}{4}g(2x)$ for all $g \in \mathbb{M}_{\tilde{\rho}}$ and all $x \in V$. Let $g, h \in \mathbb{M}_{\tilde{\rho}}$. Suppose that $\tilde{\rho}(g - h) \leq c$ for some non-negative real number c . Then we have

$$\rho(T_e g(x) - T_e h(x)) \leq \frac{1}{4}\rho(g(2x) - h(2x)) \leq \frac{L}{2}c\psi(x, 0)$$

for all $x \in V$ and so $\tilde{\rho}(T_e g - T_e h) \leq \frac{L}{2}\tilde{\rho}(g - h)$. Thus T_e is a $\tilde{\rho}$ -contraction.

Now, we claim that T_e has a bounded orbit at $\frac{1}{2}f_e$. Since f_e is an even mapping and ρ is convex, (2.7) implies that

$$(2.14) \quad \rho(f_e(2x+y) + f_e(2x-y) - f_e(x+y) - f_e(x-y) - 6f_e(x)) \leq \psi(x, y)$$

for all $x, y \in V$. Letting $y = 0$ in (2.14), we get

$$\rho(2f_e(2x) - 8f_e(x)) \leq \psi(x, 0)$$

for all $x \in V$ and so

$$\rho\left(\frac{1}{4}f_e(2x) - f_e(x)\right) \leq \frac{1}{2 \cdot 4}\psi(x, 0)$$

for all $x \in V$. For any non-negative integer n , we obtain

$$\begin{aligned} & \rho\left(\frac{1}{4^n}f_e(2^n x) - f_e(x)\right) \\ &= \rho\left(\frac{1}{2}\left[\frac{1}{2 \cdot 4^{n-1}}f_e(2^{n-1}(2x)) - \frac{1}{2}f_e(2x)\right] + \frac{1}{2}\left[\frac{1}{2}f_e(2x) - 2f_e(x)\right]\right) \\ &\leq \frac{1}{4}\rho\left(\frac{1}{4^{n-1}}f_e(2^{n-1}(2x)) - f_e(2x)\right) + \frac{1}{4}\rho(f_e(2x) - 4f_e(x)) \\ &\leq \frac{1}{4}\rho\left(\frac{1}{4^{n-1}}f_e(2^{n-1}(2x)) - f_e(2x)\right) + \frac{1}{2 \cdot 4}\psi(x, 0) \end{aligned}$$

for all $x \in V$ and by induction, we have

$$(2.15) \quad \rho\left(\frac{1}{4^n}f_e(2^n x) - f_e(x)\right) \leq \sum_{i=0}^{n-1} \frac{1}{2 \cdot 4^{i+1}}\psi(2^i x, 0)$$

for all $x \in V$ and for all non-negative integer n . Hence for any $n, m \in \mathbb{N}$, by (2.15), we get

$$\begin{aligned} & \rho\left(\frac{1}{4^n} \cdot \frac{1}{2} f_e(2^n x) - \frac{1}{4^m} \cdot \frac{1}{2} f_e(2^m x)\right) \\ & \leq \frac{1}{2} \rho\left(\frac{1}{4^n} f_e(2^n x) - f_e(x)\right) + \frac{1}{2} \rho\left(\frac{1}{4^m} f_e(2^m x) - f_e(x)\right) \\ & \leq \sum_{i=0}^{n-1} \frac{1}{4^{i+2}} \psi(2^i x, 0) + \sum_{i=0}^{m-1} \frac{1}{4^{i+2}} \psi(2^i x, 0) \leq \frac{1}{4(2-L)} \psi(x, 0) \end{aligned}$$

for all $x \in V$ and thus

$$\tilde{\rho}(T_e^n \frac{1}{2} f_e - T_e^m \frac{1}{2} f_e) \leq \frac{1}{4(2-L)}$$

for all $x \in V$. Hence T_e has a bounded orbit at $\frac{1}{2} f_e$. By Lemma 1.3, there is a $Q \in \mathbb{M}_{\tilde{\rho}}$ such that $\{T_e^n \frac{1}{2} f_e\}$ $\tilde{\rho}$ -converges to Q . Since $\tilde{\rho}$ is lower semi-continuous, we get

$$\tilde{\rho}(T_e Q - Q) \leq \liminf_{n \rightarrow \infty} \tilde{\rho}(T_e Q - T_e^{n+1} \frac{1}{2} f_e) \leq \liminf_{n \rightarrow \infty} \frac{L}{2} \tilde{\rho}(Q - T_e^n \frac{1}{2} f_e) = 0$$

and hence Q is a fixed point of T_e in $\mathbb{M}_{\tilde{\rho}}$. Replacing x and y by $2^n x$ and $2^n y$ in (2.14), respectively, we have

(2.16)

$$\begin{aligned} & \rho\left(\frac{1}{2 \cdot 4^n} [f_e(2^n(2x+y)) + f_e(2^n(2x-y)) - f_e(2^n(x+y)) \right. \\ & \left. - f_e(2^n(x-y)) - 6f_e(2^n x)]\right) \leq \frac{1}{2 \cdot 4^n} \psi(2^n x, 2^n y) \leq \frac{L^n}{2^{n+1}} \psi(x, y) \end{aligned}$$

for all $x, y \in V$. Since ρ is lower semi-continuous, by (2.16), we get

$$(2.17) \quad Q(2x+y) + Q(2x-y) - Q(x+y) - Q(x-y) - 6Q(x) = 0$$

for all $x, y \in V$. Since ρ is lower semi-continuous, by (2.15), we get

$$\rho(2Q(x) - f_e(x)) \leq \frac{1}{4(2-L)} \phi(x, 0)$$

for all $x \in X$ and so we have

$$(2.18) \quad \tilde{\rho}(2Q - f_e) \leq \frac{1}{4(2-L)}.$$

Let $F = A + Q$. Then clearly A is odd and Q is even and by (2.12) and (2.17), F is a solution of (1.1). By Theorem 2.1, F is an additive-quadratic mapping. Moreover, by (2.13) and (2.18), we have

$$\tilde{\rho}(F - \frac{1}{2} f) \leq \frac{1}{2} \tilde{\rho}(2A - f_o) + \frac{1}{2} \tilde{\rho}(2Q - f_e)$$

and hence we have (2.8).

To prove the uniqueness of F , let $G : V \rightarrow X_\rho$ be another additive-quadratic mapping with (2.8). By (2.8), we get

$$\begin{aligned} \rho\left(\frac{1}{2}G(x) - \frac{1}{2}F(x)\right) &\leq \frac{1}{2}\rho\left(G(x) - \frac{1}{2}f(x)\right) + \frac{1}{2}\rho\left(F(x) - \frac{1}{2}f(x)\right) \\ &\leq \frac{3 - 2L}{8(1 - L)(2 - L)}\psi(x, 0) \end{aligned}$$

for all $x \in V$ and so

$$\begin{aligned} \rho\left(\frac{1}{2}G_o(x) - \frac{1}{2}F_o(x)\right) &\leq \frac{1}{2}\rho\left(\frac{1}{2}G(x) - \frac{1}{2}F(x)\right) + \frac{1}{2}\rho\left(\frac{1}{2}G(-x) - \frac{1}{2}F(-x)\right) \\ &\leq \frac{3 - 2L}{8(1 - L)(2 - L)}\psi(x, 0) \end{aligned}$$

for all $x \in V$. Since F_o and G_o are fixed points of T_o , we have

$$\begin{aligned} \rho\left(\frac{1}{2}G_o(x) - \frac{1}{2}F_o(x)\right) &\leq \rho\left(\frac{1}{2}T_n G_o(x) - \frac{1}{2}T_n F_o(x)\right) \\ &\leq \frac{3 - 2L}{8(1 - L)(2 - L)}L^n\psi(x, 0) \end{aligned}$$

for all $x \in V$ and for all $n \in \mathbb{N}$. Hence $F_o = G_o$ and similarly, we have $F_e = G_e$. Thus $F = G$. \square

Using Theorem 2.2, we conclude the following classical generalized Hyers-Ulam stability in normed spaces.

COROLLARY 2.3. *Let V be a linear space, $(X, \|\cdot\|)$ a Banach space and $f : V \rightarrow X$ a mapping with $f(0) = 0$. Suppose that the following inequality*

$$\begin{aligned} &\|f(2x + y) + f(2x - y) - f(x + y) - f(x - y) - 4f(x) - f(-x)\| \\ &\leq \|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p} \end{aligned}$$

holds for all $x, y \in V$ and for some real number p with $0 < p < \frac{1}{2}$. Then there is a unique additive-quadratic mapping $F : V \rightarrow X$ such that

$$\|F(x) - f(x)\| \leq \frac{3 - 2^{2p}}{2(2 - 2^{2p})(4 - 2^{2p})}\|x\|^{2p}$$

for all $x \in X$.

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